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QUANTIZED DATA–BASED DISTRIBUTED CONSENSUS UNDER DIRECTED TIME-VARYING COMMUNICATION TOPOLOGY*

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Abstract. Distributed average consensus (DAC) is investigated for multiagent systems (MASs) with directed time-varying communication topology and quantized communication data. We propose a communication feedback–based distributed consensus protocol suitable for directed time-varying topologies to deal with the inconsistency between the internal state of each agent's encoder and the output of its neighbors' decoder, and give rigorous analysis for the consensus of the MAS. The consensus protocols are designed based on uniform quantizers with scaling. A finite lower bound of the communication data rate between each pair of adjacent agents is obtained to ensure the exponential consensus by properly choosing system parameters. In addition, the lower bound is proved to be merely 1-bit for the directed fixed topology case, no matter how large the agent number is. A numerical example is presented to demonstrate the results obtained.

Key words. multiagent system, consensus, distributed estimation, quantization, sensor network

AMS subject classifications. 68W15, 93C55, 93A14

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1. Introduction.

1.1. Motivations and related works.

Recently, distributed control and estimation of multiagent systems (MASs) have gained increased attention among researchers [14, 22, 19, 21, 8, 27, 24]. One common feature of these problems is the distributed structure constraint on the communication network; that is, each agent can use only the local communications with its neighbors and the dynamic properties of itself to adjust its behavior or modify its computation. Distributed average consensus (DAC) is one of the basic problems in this area and, based on local information, is aimed at determining how to design a protocol to ensure that all the states of the agents converge to a common value. DAC has been applied to many practical areas, such as flocking [21], formation control [8], distributed computation [26], sensor information fusion [27, 24], etc.

Most of the works listed above assume that the communication channels involved are ideal, i.e., that the communications among agents are error-free. But, in practice it is hard to avoid constraints on the communication channels, such as additive communication noises [13, 18], packet losses [7], and energy and bandwidth limitations [6, 17]. Sometimes, only integer-valued information instead of the real number sequence is required to be transmitted. This needs information quantization, and the energy and bandwidth constraints limit the communication capacity of the channels. Thus, how to realize a distributed consensus with a limited communication rate and quantized information is of great importance.

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In the control theory field, results on distributed quantized consensus have been gradually increasing recently [16, 20, 6, 5, 9, 3, 2, 15, 17, 4]. The paper [16] considered the MASs with integer-valued states, fixed topology, and an undirected connected communication graph, and gave a quantized gossip averaging algorithm, which makes the agents' states converge to an integer approximation of the initial state average with an error no greater than 1. Nedic et al. [20] considered systems with directed time-varying topologies. Based on quantized information, they developed a distributed consensus algorithm and obtained the relationship between the consensus error and the system parameters. However, when the number of the agents increases to infinity, to achieve a DAC the quantization levels are required to be infinite. Papers [5, 9] proposed an average-preserving quantized consensus protocol by using a uniform quantizer for MASs with directed fixed communication topology. Under the condition that each uniform quantizer have infinite quantization levels, it was shown that the states of all the agents converge to a neighborhood of the initial state average. The relationship between the consensus error and the system parameters was also analyzed. Paper [3] further developed an algorithm based on dynamic uniform quantization but did not provide a theoretical analysis on either the convergence of the algorithm or the impact of the quantization level on the convergence. Inspired by the work [25, 10] of stabilizing a single linear time-invariant system with minimal communication bit rate (channel capacity) for the undirected fixed topology case, [17] answered the following basic question: To ensure an MAS is consensus, how many bits of information does each pair of adjacent agents need to exchange at each time step? It is shown that for any uniform quantizer with finite quantization levels, one can always get an average consensus with an exponential convergence rate by properly choosing the system parameters.

Although there is a substantial body of works on DAC based on quantized communication data as stated in the above literatures, many problems are still unsolved. For instance, in practice, the communication connectivity between agents may change dynamically, due to the external disturbances from environments and the communication or sensing range limitations of the agents [23]. Thus, it is important to investigate how to design a DAC protocol suitable for quantized communication data and time-varying communication topologies such that an exponentially convergent consensus can be achieved even with a finite communication data rate.

1.2. Contribution of this paper. In this paper, the DAC problem is investigated for MASs with directed time-varying topologies and quantized communication data. The dynamic of each agent is described by a first-order difference equation, whose state is real-valued. The information that each agent transmits to or receives from its neighbors is integer-valued and obtained by quantizing the real-valued state via a noise-free uniform quantizer. The main contribution of the paper is summarized as follows.

A DAC protocol suitable for time-varying topologies is developed. In contrast to the undirected fixed topology case in [17], in order to keep the average of all agents' states unchanged, similar to [4], here a communication feedback channel is introduced into the dynamic encoder-decoder scheme, with which the agent can know whether or not its neighbors have received the signals whenever they are sent. It is shown that under the protocol designed, for a quite general class of time-varying topologies and any uniform quantizers with quantization levels bigger than certain finite constant, the MASs can achieve consensus exponentially provided the gain parameter and the scaling function are properly chosen. In addition, for the directed fixed topology case, it is shown that if the topology is balanced and with a spanning tree, then for any given uniform quantizer with finite levels an exponential average consensus can be achieved by properly choosing the gain parameter and the scaling function. Thus, by properly choosing the system parameters, a 1-bit quantizer for each agent can ensure consensus of the whole MAS no matter how large the agent number is.

1.3. Organization and notation. The remainder of this paper is organized as follows. In section 2, we present some notation on graph theory and describe the problem to be studied. In section 3, we discuss the DAC problem based on finite bit-rate communications, including the design of the consensus protocol and the exponential consensus analysis. In section 4, we illustrate the results via a numerical example. In section 5, we give some concluding remarks and discuss future works.

Below is a table of the basic notation to be used throughout this paper:

I_n	the n dimensional identity matrix.
$\mathbf{1}_n$	an n dimensional vector whose elements are all ones.
$\\|X\\|_\infty$	the ∞-norm of the matrix X.
$A \odot B$	the Hadamard product of the two matrices A and B.
$\lfloor a \rfloor$	the maximum integer less than or equal to the positive number a.
$\lceil a \rceil$	the minimum integer greater than or equal to the positive number a.
$b \mid_j$	the jth entry of the vector $b \in \mathbb{R}^N$.
$x_i(t)$	the state of agent i.
$\xi_{ji}(t)$	the internal state of the encoder Φ_{ji}.
$\Delta_{ji}(t)$	the output of the encoder Φ_{ji}.
$\hat{x}_{ji}(t)$	the output of the decoder Ψ_{ij}.
h	the gain parameter.
γ	the exponentially decreasing rate of the scaling function $g(t)$.
$\mathcal{G}$	the directed communication topology graph.
$\mathcal{L}_\mathcal{G}$	the Laplacian matrix of $\mathcal{G}$.
N_i^+	the in-neighbors of agent i.
N_i^-	the out-neighbors of agent i.

2. Preliminaries and problem formulation. In this section, we first introduce some preliminary notation on graph theory to be used throughout the paper, and then give the formulation of the DAC problem over digital communication channels.

2.1. Preliminaries on graph theory. Consider an MAS with N agents under a fixed communication topology. The communications among agents are modeled by a weighted digraph (communication graph) $\mathcal{G} = \{\mathcal{V}, \mathcal{E}_\mathcal{G}, \mathcal{A}_\mathcal{G}\}$ which contains a node set $\mathcal{V} = \{1, \ldots, N\}$ and an edge set $\mathcal{E}_\mathcal{G} \subseteq \mathcal{V} \times \mathcal{V}$. A node $i \in \mathcal{V}$ represents the agent i, and a directed edge $(i, j) \in \mathcal{E}_\mathcal{G}$ if and only if there is a communication link from i to j, where i is defined as the parent node, and j is defined as the child node. Here, we assume there is no self-edge (i, i) in the graph, i.e., $(i, i) \notin \mathcal{E}_\mathcal{G}$.

$\mathcal{A}_\mathcal{G} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix of $\mathcal{G}$ with $a_{ij} \geq 0$, where $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}_\mathcal{G}$. For node i, $N_i^+ = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}_\mathcal{G}\}$ and $N_i^- = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}_\mathcal{G}\}$ denote its in-neighbors and out-neighbors, respectively. The Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L}_\mathcal{G} = \mathcal{D}_\mathcal{G} - \mathcal{A}_\mathcal{G}$, where $\mathcal{D}_\mathcal{G} = \mathrm{diag}\{\sum_{j \in N_i^+} a_{ij}\}$. The degree of $\mathcal{G}$ is defined as $d^* = \max_{i \in \mathcal{V}}\{\sum_{j=1}^N a_{ij}\}$.

$\mathcal{G}$ is called a balanced graph if $\sum_{j \in N_i^+} a_{ij} = \sum_{j \in N_i^-} a_{ji}$ for all i. $\mathcal{G}$ is called an undirected graph if $\mathcal{A}_\mathcal{G}$ is symmetric. For an undirected graph $\mathcal{G}$, the eigenvalues of $\mathcal{L}_\mathcal{G}$ are denoted by $0 = \lambda_1(\mathcal{L}_\mathcal{G}) \leq \lambda_2(\mathcal{L}_\mathcal{G}) \leq \cdots \leq \lambda_N(\mathcal{L}_\mathcal{G})$. $\lambda_2(\mathcal{L}_\mathcal{G})$ is called the algebraic connectivity of $\mathcal{G}$. The mirror graph of the directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}}\}$ is an undirected graph, denoted by $\hat{\mathcal{G}} = \{\mathcal{V}, \mathcal{E}_{\hat{\mathcal{G}}}, \mathcal{A}_{\hat{\mathcal{G}}}\}$, with the same node set $\mathcal{G}$, edge set $\mathcal{E}_{\hat{\mathcal{G}}} = \mathcal{E}_{\mathcal{G}} \cup \tilde{\mathcal{E}}_{\mathcal{G}}$, and symmetric adjacency matrix $\mathcal{A}_{\hat{\mathcal{G}}} = [\hat{a}_{ij}]$, where $\tilde{\mathcal{E}}_{\mathcal{G}}$ is the reverse edge set of $\mathcal{G}$ obtained by reversing the order of nodes of all the pairs in $\mathcal{E}_{\mathcal{G}}$, and $\hat{a}_{ij} = \hat{a}_{ji} = \frac{a_{ij}+a_{ji}}{2}$ [22].

A sequence of edges (i_1, i_2), (i_2, i_3), ..., (i_{k-1}, i_k) is called a path from i_1 to i_k. The graph is called strongly connected if for any $i, j \in \mathcal{V}$ there is a path from i to j. A directed tree is a digraph, where each node except the root has exactly one parent. A spanning tree of $\mathcal{G}$ is a directed tree whose node set is $\mathcal{V}$ and whose edge set is a subset of $\mathcal{E}_{\mathcal{G}}$.

As for the time-varying communication topology case, the interactions between different agents at time t are modeled by a directed communication graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}(t)}, \mathcal{A}_{\mathcal{G}(t)}\}$, and $\mathcal{E}_{\mathcal{G}(t)}$, $\mathcal{A}_{\mathcal{G}(t)}$, $\mathcal{D}_{\mathcal{G}(t)}$, $\mathcal{L}_{\mathcal{G}(t)}$, $N_i^+(t)$, $N_i^-(t)$, $d^*(t)$ have the same meanings as in the fixed topology case, for instance, $d^*(t) = \max_{i\in\mathcal{V}}\{\sum_{j=1}^N a_{ij}(t)\}$. For a given positive integer k, the union of k digraphs $\{\mathcal{G}(i), i = 1, \ldots, k\}$ is denoted by $\sum_{i=1}^k \mathcal{G}(i) = \left\{\mathcal{V}, \cup_{i=1}^k \mathcal{E}_{\mathcal{G}(i)}, \sum_{i=1}^k \mathcal{A}_{\mathcal{G}(i)}\right\}$.

2.2. Problem formulation. In this paper, we consider the DAC problem for an MAS with N agents, using quantized communication data. The dynamics of each agent is described by the following first-order difference equation:

$$x_i(t+1) = x_i(t) + hu_i(t), \quad t = 0, 1, \ldots, \quad i = 1, \ldots, N, \tag{2.1}$$

where $x_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ denote the state and control of the ith agent, respectively; h is the gain parameter.

We assume each pair of adjacent agents use a digital communication channel to exchange symbol information. Thus, the real-valued state of each agent should be quantized first before it is transmitted. In this paper, the communication scheme between each pair of adjacent agents consists of a dynamic encoder-decoder pair and an unreliable digital communication channel. After encoding its real-valued state, the agent i will send its encoder's internal state to its ideal out-neighbors N_i^-. Since the communication link is unreliable, the out-neighbors are time-varying with $N_i^-(t) \subseteq N_i^-$, which makes the communication graph time-varying. We denote the communication graph at time t by $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}(t)}, \mathcal{A}_{\mathcal{G}(t)}\}$ with $\mathcal{E}_{\mathcal{G}(t)} \subseteq \mathcal{E}_{\mathcal{G}}$, where $\mathcal{G} = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}}\}$ denotes the ideal communication network of agents without link failures. The encoder and decoder are designed based on the uniform quantizer $q(\cdot) : \mathbb{R} \to \Lambda = \{0, \pm i, i = 1, \ldots, K\}$,

$$q(y) = \begin{cases} 0, & -1/2 \le y < 1/2, \\ i, & (2i-1)/2 \le y < (2i+1)/2, \\ & i = 1, \ldots, K-1, \\ K, & y \ge (2K-1)/2, \\ -q(-y), & y \le -1/2, \end{cases} \tag{2.2}$$

where Λ denotes the set of quantization levels, and K is a positive integer. In this case, the number of quantization levels is $2K + 1$.

Based on the above communication scheme and system dynamics (2.1), the DAC problem is to design a control $u_i(t)$ for each agent i ($i \in \mathcal{V}$) using only local quantized information to make the states of all the agents converge asymptotically to $f(X(0)) = \frac{1}{N}\sum_{i=1}^N x_i(0)$, where $X(0) = [x_1(0), \ldots, x_N(0)]^T$.

For the above DAC problem, in the following section we mainly focus on how to design consensus protocols, how to obtain the explicit lower bound of quantization levels, and how to prove the exponential consensus of the whole agent system by properly choosing system parameters.

3. DAC under quantized communication data. This section is devoted to the DAC under time-varying communication topology by using finite bit-rate communications. To this end, we first give the communication scheme to be adopted, and then construct a distributed control based on quantized communication data. Finally, consensus properties of the agent system are analyzed. The special case under directed fixed topology is also discussed.

Compared with the communication scheme under the undirected fixed communication topology case in [17], the time-varying topology results in the following problem: When agent j encodes its state $x_j(t)$ and transmits the encoded data to its ideal neighbor agent $i \in N_j^-$, the transmission may fail due to the uncertainty of the communication channel, which causes the decoder of agent i to not update its state estimate of agent j, and causes the decoder's output to not be the same as the internal state of agent j's encoder. Thus, different from the fixed topology case, the internal state of the encoder of agent j is usually unknown to its neighbor $j \in N_i^-$. To solve this problem, we need to redesign the error-compensation–type protocol to make it adapt to the time-varying communication topologies. The key idea is to construct a suitable encoder-decoder scheme such that both the sender and receiver agent can obtain the same estimate of the sender's state even when the communication graph is time-varying. Based on the state estimates known by both the sender and receiver, the error-compensation approach [17] can then be applied to design the distributed consensus protocol. Below, we first present the design of the encoder-decoder scheme, and then give the formal statement of the consensus control.

At the sender side of the channel $(j,i) \in \mathcal{E}_{\mathcal{G}}$, agent j $(j = 1,\ldots,N)$ encodes its state by the encoder Φ_{ji} and sends the output of the encoder to its out-neighbor $i \in N_j^-$. The encoder $\Phi_{ji} \in \Phi_j \triangleq \{\Phi_{ji} : i \in N_j^-\}$ is defined by

$$(3.1)\qquad \begin{aligned} &\xi_{ji}(0) = 0,\\ &\Delta_{ji}(t) = q\left(g^{-1}(t-1)(x_j(t) - \xi_{ji}(t-1))\right),\\ &\xi_{ji}(t) = \begin{cases} g(t-1)\Delta_{ji}(t) + \xi_{ji}(t-1) & \text{if } \Delta_{ji}(t) \text{ is received by } i \text{ at time } t,\\ \xi_{ji}(t-1) & \text{otherwise}, \quad t = 1,2,\ldots, \end{cases} \end{aligned}$$

where $\xi_{ji}(t)$ is the internal state of Φ_{ji}, $\Delta_{ji}(t)$ is the output of Φ_{ji} to be sent to the neighbor agent i, $g(t) > 0$ is a scaling function, and $q(\cdot)$ is the uniform quantizer defined in (2.2) with $2K + 1$ quantization levels. In this case, the communication channel $(j,i) \in \mathcal{E}_{\mathcal{G}}$ is required to be capable of transmitting $\lceil \log_2(2K) \rceil$ bits of data without error at each time step.

At the receiver side of the channel $(j,i) \in \mathcal{E}_{\mathcal{G}}$, agent $i \in N_j^-$ updates the output of its decoder Ψ_{ij} according to whether or not it receives $\Delta_{ji}(t)$ and estimates the state $x_j(t)$. The decoder $\Psi_{ij} \in \Psi_i \triangleq \{\Psi_{ij} : j \in N_i^+\}$ is defined by

$$(3.2)\qquad \begin{aligned} &\hat{x}_{ji}(0) = 0,\\ &\hat{x}_{ji}(t) = \begin{cases} g(t-1)\Delta_{ji}(t) + \hat{x}_{ji}(t-1) & \text{if } \Delta_{ji}(t) \text{ is received by } i \text{ at time } t,\\ \hat{x}_{ji}(t-1) & \text{otherwise}, \quad t = 1,2,\ldots, \end{cases} \end{aligned}$$

where $\hat{x}_{ji}(t)$ is the output of the decoder Ψ_{ij} at time t. From (3.1) and (3.2), the same recursive definition of $\xi_{ji}(t)$ and $\hat{x}_{ji}(t)$ with both zero initial values ensures $\xi_{ji}(t) = \hat{x}_{ji}(t)$, $t \geq 0$. Thus, we have constructed the same estimate of each sender's state at both the sender and receiver sides.

Based on the above dynamic encoder-decoder scheme, under the time-varying communication topology sequence $\{\mathcal{G}(t), t = 0, 1, \ldots\}$ we can construct the following distributed error-compensation–type consensus protocol:

$$u_i(t) = \sum_{j \in N_i^+(t)} a_{ij}(t)\hat{x}_{ji}(t) - \sum_{j \in N_i^-(t)} a_{ji}(t)\xi_{ij}(t), \quad i = 1, 2, \ldots, N. \tag{3.3}$$

Remark 3.1. To apply the consensus protocol (3.3), each agent needs to know the following information: the neighbor link weights, the output of its encoders and its in-neighbors' encoders, and the scaling gain function. From the discussion below, the scaling gain function $g(t)$ is designed off-line, which requires knowledge of the upper bound of the initial states, the upper bound of the norm of the Laplacian matrices, and the positive lower bound of the algebraic connectivity of the mirror graphs.

Remark 3.2. The dynamic encoder-decoder pair (3.1), (3.2) is a difference coding algorithm with scaling, where the "prediction error" $x_j(t) - \xi_{ji}(t-1)$ is quantized rather than the state $x_j(t)$. Generally, such difference coding schemes have advantages in saving communication bits. Similar difference coding algorithms can be found in [3], where the zoom-out and zoom-in constant parameters are used to adjust the prediction error according to whether or not it exceeds the quantization domain. In contrast, here we will design the scaling function $g(t)$ off-line by properly choosing a constant parameter. Other coding schemes that use a constant quantization step-size to quantize the state directly can be found in [9, 16, 20].

Remark 3.3. One difficulty in the encoder-decoder pair (3.1), (3.2) is that agent j needs to know whether or not its encoder's output has been received by its out-neighbor $i \in N_j^-$. This problem can be solved by adding a communication feedback as in [4]. In fact, by using a noise-free communication feedback channel, agent i can send back a bit signal "1" to tell agent j that it has received $\Delta_{ji}(t)$, or send back a bit signal "0" to tell agent j that it did not receive $\Delta_{ji}(t)$, or not do anything and let agent j itself decide after a certain time period whether or not its quantized signal has been received by agent i. In the undirected topology case, the above communication feedback scheme is good enough for the realization of DAC, but in the directed topology case, we have to use this feedback channel to let agent i know the weight $a_{ji}(t)$ of its neighbor $j \in N_i^-(t)$, which is not needed in the undirected topology case.

Remark 3.4. In consensus protocol (3.3), dynamic properties of the interaction topologies $\{\mathcal{G}(t), t \geq 0\}$ consist of two aspects. One is the number of each agent's neighbors, and the other is the weight of each edge, which represents the variations of relative reliability of each communication link at different times. By definition, when $i \in N_j^+(t)$, we have $a_{ij}(t) > 0$.

Let

$$\begin{aligned} X(t) &= [x_1(t), \ldots, x_N(t)]^T, \\ \delta(t) &= X(t) - J_N X(t), \quad J_N = \frac{1}{N}\mathbf{1}\mathbf{1}^T. \end{aligned} \tag{3.4}$$

From (2.1) and (3.3), the closed-loop system can be described in the following compact form:

$$X(t+1) = \left(I - h\mathcal{L}_{\mathcal{G}(t)}\right) X(t) - h\left[\left(\mathcal{L}_{\mathcal{G}(t)} \odot \Lambda(t)\right) - \left(\mathcal{L}_{\mathcal{G}(t)} \odot \Lambda(t)\right)^T\right]\mathbf{1}, \tag{3.5}$$

where $X(t)$ is defined as in (3.4), $\Lambda(t) = [\lambda_{ij}(t)]$, and $\lambda_{ij}(t)$ is defined by

$$\lambda_{ij}(t) = \begin{cases} \hat{x}_{ji}(t) - x_j(t), & j \in N_i^+, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Below, we concentrate on analyzing the consensus properties of (3.5). The key point is to prove that the quantizer (3.1), (3.2) is never saturated. Intuitively, this can be ensured by choosing the decaying rate of the scaling function $g(t)$ smaller than that of the consensus algorithm without quantization. To this end, we make the following assumptions on the directed time-varying communication graph sequence and the initial states:

(A1) $\{\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}(t)}, \mathcal{A}_{\mathcal{G}(t)}\}, t = 0, 1, \dots\}$ is a balanced digraph sequence, and there exist an integer $h_0 > 0$ and a constant $\lambda_0 > 0$ such that $\inf_{m\geq 0} \lambda_{mh_0}^{h_0} \geq \lambda_0$, where $\lambda_k^{h_0} = \lambda_2(\mathcal{L}_{\hat{\mathcal{G}}_k^{h_0}})$, and $\mathcal{G}_k^{h_0} = \sum_{i=k}^{k+h_0-1} \mathcal{G}(i)$, and $\hat{\mathcal{G}}_k^{h_0}$ is the mirror graph of $\mathcal{G}_k^{h_0}$.

(A2) There is a positive integer T_0, such that for any time instant $t_1 \geq 0$ and any agent $j \in N_i^+$, $i = 1, \dots, N$, $j \in N_i^+(t)$ holds at least once in $[t_1, t_1 + T_0)$.

(A3) $\max_i |x_i(0)| \leq C_x$, $\max_i |\delta_i(0)| \leq C_\delta$, where C_x and C_δ are known nonnegative constants.

Remark 3.5. Different from the connectivity condition on the gossip communication protocol used in [1], assumption (A1) does not need additional conditions on the distribution of the random communication graph sequence. From [18, Lemma 4.1], (A1) is equivalent to the periodical connectivity condition: there is a positive integer h_0 such that for any $t \geq 0$, $\sum_{k=t}^{t+h_0-1} \mathcal{G}(k)$ contains a spanning tree. A periodical connectivity condition is often used in the literatures of DAC over time-varying topologies [14, 20]. Intuitively, it guarantees the existence of a finite time period such that for any pair of agents i, j, starting from any time instant t, agent i can always influence agent j in this time period only by local interactions among agents. This condition is not necessary for the convergence of the distributed consensus algorithm. For instance, under the ultimate connectivity condition, [19, 23] proved the convergence of the distributed consensus protocol for the case of ideal communication data and an undirected time-varying topology. However, we can show that assumption (A1) is a sufficient and necessary condition to ensure an exponential convergence of the distributed consensus protocol for the case of ideal communication data and a directed balanced time-varying topology. Actually, for system (2.1) and the consensus protocol with ideal communication data

$$u_i(t) = \sum_{j \in N_i^+(t)} a_{ij}(t) \left(x_j(t) - x_i(t)\right), \quad i = 1, 2, \dots, N,$$

we make the state transform $y(t) = [y_1(t), \dots, y_N(t)]^T = \Psi^{-1} x(t)$, where $\Psi = [\frac{1}{\sqrt{N}}\mathbf{1}, \Psi_{N\times(N-1)}]$, $\Psi_{N\times(N-1)} \in \mathbb{R}^{N\times(N-1)}$ satisfies $\mathbf{1}^T \Psi_{N\times(N-1)} = 0$, $\Psi_{N\times(N-1)}^T \Psi_{N\times(N-1)} = I$. Then, we have

$$\begin{aligned} y_1(t+1) &= y_1(t), \\ y^{(N-1)}(t+1) &= \left(I - h\Psi_{N\times(N-1)}^T \mathcal{L}_{\mathcal{G}(t)} \Psi_{N\times(N-1)}\right) y^{(N-1)}(t), \end{aligned}$$

where $y^{(N-1)}(t) = [y_2(t), \dots, y_N(t)]^T$. Define the exponential consensus factor [3]:

$$\begin{aligned} \rho &\triangleq \limsup_{t\to\infty} \|X(t) - J_N X(0)\|^{\frac{1}{t}} = \limsup_{t\to\infty} \|X^T(t)(I - J_N)X(t)\|^{\frac{1}{2t}} \\ &= \limsup_{t\to\infty} \|y^{(N-1)^T}(t) y^{(N-1)}(t)\|^{\frac{1}{2t}}. \end{aligned}$$

Then, to prove that the closed-loop system achieves exponential consensus, it is equivalent to prove that $y^{(N-1)}(t)$ converges to 0 exponentially. Choose sufficiently small h such that all eigenvalues of the nonnegative matrix $h\Psi^T_{N\times(N-1)}\mathcal{L}_{\mathcal{G}(t)}\Psi_{N\times(N-1)}$ are no greater than 1. Then, from [12, Theorem 2.3.2], a sufficient and necessary condition for $y^{(N-1)}(t)$ converging exponentially to 0 is that there is a positive integer h_0 such that

$$\begin{aligned}&\inf_{t\geq 0}\lambda_{\min}\left\{\Psi^T_{N\times(N-1)}\left(\sum_{k=t+1}^{t+h_0}\mathcal{L}_{\mathcal{G}(k)}\right)\Psi_{N\times(N-1)}\right\}\\ &\quad=\inf_{t\geq 0}\lambda_{\min}\left\{\Psi^T_{N\times(N-1)}\mathcal{L}_{\sum_{k=t+1}^{t+h_0}\mathcal{G}(k)}\Psi_{N\times(N-1)}\right\}\neq 0,\end{aligned}$$

which is equivalent to assumption (A1).

Remark 3.6. In the case of the finite communication data rate, assumption (A2) ensures that the encoder (3.1) and decoder (3.2) of each agent are not saturated. If the quantization levels of each agent are high enough and ensure that the dynamic encoder and decoder are not saturated, then we can get the consensus convergence even without assumption (A2), but we cannot achieve the quantitative relationship between the communication data rate and the associated system parameters.

Remark 3.7. Assumption (A3) gives an upper bound for the initial states and initial consensus errors. In fact, the existence of C_x also implies the existence of C_δ; that is, C_δ can be taken as $2C_x$. However, in many cases, the upper bound of the initial consensus error may be much smaller than $2C_x$. If the upper bound of the initial consensus error is used to estimate system coefficients, then more accurate estimates of the associated coefficients are expected to be achieved. Thus, we use a separate constant to denote the upper bound of the initial consensus error.

We now study the convergence property of the closed-loop system (3.5).

THEOREM 3.8. *Suppose assumptions* (A1)–(A3) *hold. For given positive constants h, ϵ_1, ϵ_2, γ, and g_0, let*

$$\begin{aligned}K_1(h,\gamma,\epsilon_1,\epsilon_2)&:=\left\lfloor M_1(h,\gamma,\epsilon_1,\epsilon_2)-\frac{1}{2}\right\rfloor+1,\\ M_1(h,\gamma,\epsilon_1,\epsilon_2)&:=\max\left\{\frac{1}{2\gamma},M^{\frac{1}{2}}+M^{\frac{1}{2}}\gamma^{-(T_0+1)}+\frac{1}{2}\gamma^{-(T_0+2)}\right\}+\frac{hd^*}{\gamma}+2hd^*M^{\frac{1}{2}},\\ M:=M(h,\gamma,\epsilon_1,\epsilon_2)&=\frac{N\rho_{h,\epsilon_1}{d^*}^2\rho_{h,\epsilon_2}^{h_0}}{\epsilon_1\gamma^4(\gamma^{2h_0}-\rho_{h,\epsilon_1})}\sum_{j=0}^{2h_0-2}\left(h_0-|j-(h_0-1)|\right)\gamma^{j-2}\\ &\quad\cdot\sum_{l=0}^{j}C_j^lh^lL^l+\frac{N\rho_{h,\epsilon_2}{d^*}^2(1-\gamma^{-4h_0}\rho_{h,\epsilon_2}^{2h_0})}{\epsilon_2\gamma^4(1-\gamma^{-2}\rho_{h,\epsilon_2})},\\ \rho_{h,\epsilon_1}&:=1-2h\lambda_0+\sum_{l=2}^{2h_0}h^lC_{2h_0}^lL^l+\epsilon_1h^2,\quad \rho_{h,\epsilon_2}:=1+2hL+h^2L^2+\epsilon_2h^2,\\ \Theta_1&:=N\rho_{h,\epsilon_2}^{2h_0}\gamma^{-4h_0}(\gamma^{2h_0}-\rho_{h,\epsilon_1})(C_\delta^2+4\rho_{h,\epsilon_2}\epsilon_2^{-1}\gamma^{-2}{d^*}^2C_x^2),\\ \Theta_2&:=N\rho_{h,\epsilon_1}\epsilon_2^{-1}{d^*}^2\gamma^{-4}\rho_{h,\epsilon_2}^{h_0}\sum_{j=0}^{2h_0-2}\left(h_0-|j-(h_0-1)|\right)\gamma^{j-2}\sum_{l=0}^{j}C_j^lh^lL^l.\end{aligned}\tag{3.7}$$

If $\rho_{h,\epsilon_1} \in (0,1)$, $\gamma \in (\rho_{h,\epsilon_1}^{\frac{1}{2h_0}}, 1)$, *$K$ is a given positive integer such that* $K \geq K_1(h,\gamma,\epsilon_1,\epsilon_2)$, *and*

$$g_0 > \max\left\{\frac{(1+2hd^*)C_x + 2hd^*C_\delta}{K+\frac{1}{2}}, \left(\frac{\Theta_1}{\Theta_2}\right)^{\frac{1}{2}}\right\}, \tag{3.8}$$

then under the protocol (3.1)–(3.3) *with the* $(2K+1)$*-level uniform quantizer* (2.2) *and the scaling function* $g(t) = g_0\gamma^t$, *the closed-loop system* (3.5) *achieves consensus exponentially, i.e.,*

$$\left\| x_i(t) - \frac{1}{N}\sum_{j=1}^{N} x_j(0) \right\| = O(\gamma^t), \quad t \to \infty,\ i = 1,\ldots,N, \tag{3.9}$$

where d^* *is a positive constant satisfying* $d^* \geq \sup_k d^*(k)$, $d^*(k)$ *is the degree of* $\mathcal{G}(k)$, T_0 *is a constant denoted in* (A2), C_j^l *is the combinatorial number determined by choosing l numbers from j numbers, and* $L \geq \sup_k \|\mathcal{L}_{\mathcal{G}(k)}\|$.

Proof. See Appendix A. ∎

Remark 3.9. From Theorem 3.8, the closed-loop system can achieve average consensus exponentially. To obtain a faster convergence rate, we should make γ as close to $\rho_{h,\epsilon_1}^{\frac{1}{2h_0}}$ as possible, and choose h, ϵ_1 to make ρ_{h,ϵ_1} as small as possible. However, from (3.7) it can be seen that the bits needing to be communicated will increase as γ becomes small, and when $\gamma \to \rho_{h,\epsilon_1}^{\frac{1}{2h_0}}$, the bits will become infinite. In practice, we may have to face limitations on communication bits, which requires us to conduct the convergence analysis of the system (3.5) under a fixed finite number of quantization levels. This may result in a slow convergence rate.

THEOREM 3.10. *Suppose assumptions* (A1)–(A3) *hold. Then for any integer* $K \geq \lfloor \overline{M}_1 \rfloor + 1$, *the following parameter vector set* $\Omega_1(K)$ *is nonempty:*

$$\Omega_1(K) = \left\{(h,\gamma,\epsilon_1,\epsilon_2) \mid \rho_{h,\epsilon_1} \in (0,1), \gamma \in \left(\rho_{h,\epsilon_1}^{\frac{1}{2h_0}}, 1\right), M_1(h,\gamma,\epsilon_1,\epsilon_2) < K + \frac{1}{2}\right\},$$

where ρ_{h,ϵ_1}, $M_1(h,\gamma,\epsilon_1,\epsilon_2)$ *are defined as in* (3.7), *and*

$$\overline{M}_1 = d^* N^{\frac{1}{2}} \left(\sum_{j=0}^{2h_0-2} (h_0 - |j-(h_0-1)|)\right)^{\frac{1}{2}} \cdot \max\left\{\frac{2}{\lambda_0}, \frac{1}{L^{\frac{1}{2}}|\lambda_0\sqrt{C_{2h_0}^2} - LC_{2h_0}^2|^{\frac{1}{2}}}\right\}.$$

For any $(h^*,\gamma^*,\epsilon_1^*,\epsilon_2^*) \in \Omega_1(K)$, *under the control* (3.1), (3.2), (3.3) *with the* $(2K+1)$*-level uniform quantizer* (2.2) *and the scaling function* $g(t) = g_0{\gamma^*}^t$, *the closed-loop system* (3.5) *satisfies*

$$\lim_{t\to\infty} x_i(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(0), \quad i = 1,\ldots,N,$$

where g_0 *is a constant satisfying* (3.8).

Proof. Let κ^* and h be two constants satisfying

$$\begin{gathered}\max\left\{2\left(\lambda_0 - L\sqrt{C_{2h_0}^2}\right), 0\right\} < \kappa^* < 2\lambda_0, \\ 0 < h < \min\left\{\frac{2\lambda_0\kappa^*}{\sum_{l=1}^{2h_0-1} C_{2h_0}^{l+1} L^{l+1}}, 1\right\},\end{gathered} \tag{3.10}$$

and set $\epsilon_1 = \kappa^* h^{-1}$. Then, by the definition of ρ_{h,ϵ_1} in (3.7), we have $\rho_{h,\epsilon_1} \in (0,1)$. This means that there exist h, $\epsilon_1 > 0$ such that $\rho_{h,\epsilon_1} \in (0,1)$.

For any given $\kappa_1^* > 0$, set $\epsilon_2 = \kappa_1^* h^{-1}$. By the definition of $M(h,\gamma,\epsilon_1,\epsilon_2)$ in (3.7) we have

$$\begin{aligned}
&\lim_{h\to 0} M(h,1,\kappa^* h^{-1},\epsilon_2)\\
&\quad = \lim_{h\to 0} \frac{N {d^*}^2}{\kappa^*(2\lambda_0 - \kappa^* - \sum_{l=1}^{2h_0-1} h^l C_{2h_0}^{l+1} L^{l+1})} \sum_{j=0}^{2h_0-2} (h_0 - |j-(h_0-1)|) \sum_{l=0}^{j} C_j^l h^l L^l\\
&\qquad + \lim_{h\to 0} \frac{N {d^*}^2 h (1-\rho_{h,\epsilon_2}^{2h_0})}{\kappa_1^*(1-\rho_{h,\epsilon_2})} = \frac{N {d^*}^2}{\kappa^*(2\lambda_0 - \kappa^*)} \sum_{j=0}^{2h_0-2} (h_0 - |j-(h_0-1)|).
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
&\lim_{h\to 0}\left(2(1+hd^*) M^{\frac{1}{2}}(h,1,\kappa^* h^{-1},\epsilon_2) + hd^* + \frac{1}{2}\right)\\
&\quad = 2d^* \left(\frac{N \sum_{j=0}^{2h_0-2} (h_0 - |j-(h_0-1)|)}{\kappa^*(2\lambda_0 - \kappa^*)}\right)^{\frac{1}{2}} + \frac{1}{2}.
\end{aligned}$$

From (3.10), by properly choosing κ^*, the above equation can achieve $\overline{M}_1 + \frac{1}{2}$. Thus, for any given $K \ge \lfloor \overline{M}_1 \rfloor + 1$, there are κ^* and h^* satisfying (3.10), and $\epsilon_2^* = \kappa_1^* {h^*}^{-1}$ such that

$$2\left(1+h^* d^*\right) M^{\frac{1}{2}}(h^*,1,\kappa^* {h^*}^{-1},\epsilon_2^*) + h^* d^* + \frac{1}{2} < \lfloor \overline{M}_1 \rfloor + \frac{3}{2} \le K + \frac{1}{2}.$$

Let $\epsilon_1^* = \kappa^* {h^*}^{-1}$. Then, from the above inequality and (3.7) it follows that

$$\lim_{\gamma\to 1} M_1(h^*,\gamma,\epsilon_1^*,\epsilon_2^*) = 2(1+h^* d^*) M^{\frac{1}{2}}(h^*,1,\kappa^* {h^*}^{-1},\epsilon_2^*) + h^* d^* + \frac{1}{2} < K + \frac{1}{2}.$$

Thus, there is $\gamma^* \in (\rho_{h^*,\epsilon_1^*}^{\frac{1}{2h_0}}, 1)$ such that $M_1(h^*,\gamma^*,\epsilon_1^*,\epsilon_2^*) < K + \frac{1}{2}$. This together with Theorem 3.8 implies the theorem. ∎

Remark 3.11. Theorem 3.10 gives a finite lower bound for the quantization level so that system coefficients can be chosen to make the whole agent system achieve exponential consensus. Compared with the undirected fixed topology case in [17], here it is hard to extend the result to the case of any finite quantization level. The reason is mainly due to the accumulative effect of the encoder-decoder scheme (3.1), (3.2) designed to adapt for the time-varying topology. An additional estimation term for the quantization level naturally appears (see the first term of $M_1(h,\gamma,\epsilon_1,\epsilon_2)$ in (3.7)), which leads to the lower bound for the required quantization level.

Based on the same idea as the design of the above consensus protocol in the directed time-varying topology case, the corresponding results for the directed fixed topology case can be easily obtained. To avoid redundancy, below we will simply present the communication scheme, the distributed control, and the convergence results, illustrating only the differences between these and the time-varying case. The following assumption on the fixed communication graph $\mathcal{G}$ is needed.

(A4) $\mathcal{G}$ is directed and balanced and contains a spanning tree.

The communication scheme contains an encoder at the sender side of a noise-free digital channel $(j,i) \in \mathcal{E}_{\mathcal{G}}$, and a decoder at the receiver side of the channel (j,i). The encoder Φ_j of agent j $(j = 1, \dots, N)$ is defined by

$$(3.11)\qquad \begin{cases} \xi_j(0) &=& 0, \\ \Delta_j(t) &=& q(g^{-1}(t-1)(x_j(t) - \xi_j(t-1))), \\ \xi_j(t) &=& g(t-1)\Delta_j(t) + \xi_j(t-1), \quad t = 1, 2, \dots, \end{cases}$$

and the decoder Ψ_i at the receiver side of agent i is defined by

$$(3.12)\qquad \begin{cases} \hat{x}_{ji}(0) &=& 0, \\ \hat{x}_{ji}(t) &=& g(t-1)\Delta_j(t) + \hat{x}_{ji}(t-1), \end{cases}$$

where $\xi_j(t)$ is the internal state of Φ_j; $\Delta_j(t)$ is the output of Φ_j, which will be sent to the out-neighbors of agent j; $\hat{x}_{ji}(t)$ is the output of Ψ_i; and $g(t) > 0$ is a scaling function. From (3.11) and (3.12), the same recursive expression and initial values of $\{\xi_j(t), t \geq 0\}$ and $\{\hat{x}_{ji}(t), t \geq 0\}$ make $\xi_j(t) = \hat{x}_{ji}(t)$, which is key to the design of the following error-compensation consensus protocol.

For the fixed topology case, the error-compensation–type average consensus protocol (3.3) becomes

$$(3.13)\qquad u_i(t) = \sum_{j \in N_i^+} a_{ij}(\hat{x}_{ji}(t) - \xi_i(t)), \quad t = 0, 1, \dots, \; i = 1, 2, \dots, N,$$

and the compact closed-loop system by substituting (3.11)–(3.13) into (2.1) is

$$(3.14)\qquad \begin{cases} X(t+1) &=& (I - h\mathcal{L}_{\mathcal{G}})\, X(t) + h\mathcal{L}_{\mathcal{G}} e(t), \\ \hat{X}(t+1) &=& g(t)Q\left[g^{-1}(t)\left(X(t+1) - \hat{X}(t)\right)\right] + \hat{X}(t), \end{cases}$$

where $X(t)$ is defined as in (3.4), $\hat{X}(t) = [\xi_1(t), \dots, \xi_N(t)]^T$, $e(t) = X(t) - \hat{X}(t)$, $Q([y_1, \dots, y_N]^T) = [q(y_1), \dots, q(y_N)]^T$.

Below, we will analyze the convergence property of the closed-loop system (3.14). Similar to Theorem 3.8, the key point is to prove that the quantizer (3.11), (3.12) is never saturated. Intuitively, this can be ensured by choosing the decaying rate of the scaling function $g(t)$ smaller than that of the consensus algorithm without quantization. The results corresponding to Theorems 3.8 and 3.10 can be summarized by the following two theorems.

THEOREM 3.12. *Suppose assumptions* (A3) *and* (A4) *hold. For given positive constants h, γ, ϵ_3, and g_0, let*

$$(3.15)\qquad \begin{gathered} K_2(h,\gamma,\epsilon_3) := \left\lfloor M_2(h,\gamma,\epsilon_3) - \frac{1}{2} \right\rfloor + 1, \\ M_2(h,\gamma,\epsilon_3) := \frac{1 + 2hd^*}{2\gamma} + \frac{h\sqrt{N}\overline{L}^2 \rho_{h,\epsilon_3}^{\frac{1}{2}}}{2\gamma\epsilon_3^{\frac{1}{2}}(\gamma^2 - \rho_{h,\epsilon_3})^{\frac{1}{2}}}, \\ \rho_{h,\epsilon_3} := 1 - 2h\bar{\lambda}_0 + \left(\epsilon_3 + \overline{L}^2\right) h^2. \end{gathered}$$

If $\rho_{h,\epsilon_3} \in (0,1)$, $\gamma \in (\rho_{h,\epsilon_3}^{\frac{1}{2}}, 1)$, K is a positive integer such that $K \geq K_2(h,\gamma,\epsilon_3)$, and

$$(3.16)\qquad g_0 > \max\left\{ \frac{C_x}{K + \frac{1}{2}}, 2\left[\left(\gamma^2 - \rho_{h,\epsilon_3}\right)\left(C_x^2 + \epsilon_3 C_\delta^2 \overline{L}^{-2}\right)\right]^{\frac{1}{2}} \right\},$$

then under the control (3.11)–(3.13) *with the* $(2K+1)$*-level uniform quantizer* (2.2) *and the scaling function* $g(t)=g_0\gamma^t$*, the closed-loop system* (3.14) *satisfies*

$$\left\| x_i(t)-\frac{1}{N}\sum_{j=1}^{N}x_j(0)\right\| = O(\gamma^t) \quad \text{as } t\to\infty, \quad i=1,\dots,N, \tag{3.17}$$

where $\bar{\lambda}_0$, $\overline{L}$ *are positive constants satisfying* $0<\bar{\lambda}_0\le\lambda_2(\mathcal{L}_{\hat{\mathcal{G}}})$, $\overline{L}\ge\|\mathcal{L}_{\mathcal{G}}\|$, *and* $\hat{\mathcal{G}}$ *is the mirror graph of* $\mathcal{G}$.

Proof. See Appendix B. □

Similar to the discussion in Remark 3.9, from Theorem 3.12 it can be seen that a trade-off between the quantization level and the convergence rate also exists in the fixed topology case. However, different from the time-varying topology case, the following theorem implies that for uniform quantizers with any fixed quantization level, no matter how large the agent number N is, exponential average consensus can be achieved by properly choosing system parameters.

THEOREM 3.13. *Suppose assumptions* (A3) *and* (A4) *hold. Then, for any integer* $K\ge 1$*, the following parameter set* $\Omega_2(K)$ *is nonempty:*

$$\Omega_2(K)=\left\{(h,\gamma,\epsilon_3)\mid h,\epsilon_3>0,\rho_{h,\epsilon_3}\in(0,1),\gamma\in\left(\rho_{h,\epsilon_3}^{\frac{1}{2}},1\right),M_2(h,\gamma,\epsilon_3)<K+\frac{1}{2}\right\},$$

where ρ_{h,ϵ_3} *and* $M_2(h,\gamma,\epsilon_3)$ *are defined as in* (3.15)*. Furthermore, for any* $(h^*,\gamma^*,\epsilon_3^*)\in\Omega_2(K)$*, under the control* (3.11)–(3.13) *with the* $(2K+1)$*-level uniform quantizer* (2.2) *and the scaling function* $g(t)=g_0{\gamma^*}^t$*, the closed-loop system* (3.14) *achieves consensus, where* g_0 *is a constant satisfying* (3.16).

Proof. We first show there exist $h>0$ and $\epsilon_3>0$ such that $\rho_{h,\epsilon_3}\in(0,1)$, i.e.,

$$0<1-2h\bar{\lambda}_0+\left(\epsilon_3+\overline{L}^2\right)h^2<1. \tag{3.18}$$

Notice that when $0<h<\frac{2\bar{\lambda}_0}{\epsilon_3+\overline{L}^2}$ and $\bar{\lambda}_0^2-(\epsilon_3+\overline{L}^2)<0$, the inequality (3.18) is satisfied. Then, for all $\epsilon_3^*>\max\{\bar{\lambda}_0^2-\overline{L}^2,0\}$ and $0<h<\frac{2\bar{\lambda}_0}{\epsilon_3^*+\overline{L}^2}$, we have $\rho_{h,\epsilon_3^*}\in(0,1)$.

From

$$\lim_{h\to 0}\left\{h\sqrt{N}\overline{L}^2\left(\frac{h^2+{\epsilon_3^*}^{-1}\left(1-2h\bar{\lambda}_0+h^2\overline{L}^2\right)}{8h\bar{\lambda}_0-4h^2(\epsilon_3^*+\overline{L}^2)}\right)^{\frac{1}{2}}+\frac{1+2hd^*}{2}\right\}$$

$$=\lim_{h\to 0}\left\{\sqrt{h}\sqrt{N}\overline{L}^2\right\}\cdot\lim_{h\to 0}\left(\frac{{\epsilon_3^*}^{-1}+h\left(h-2\bar{\lambda}_0{\epsilon_3^*}^{-1}+h\overline{L}^2\right)}{8\bar{\lambda}_0-4h\left(\epsilon_3^*+\overline{L}^2\right)}\right)^{\frac{1}{2}}+\lim_{h\to 0}\{hd^*\}+\frac{1}{2}=\frac{1}{2},$$

one can see that for any given $K\ge 1$, there is $h^*\in(0,\frac{2\bar{\lambda}_0}{\epsilon_3^*+\overline{L}^2})$ such that

$$h^*\sqrt{N}\overline{L}^2\left(\frac{{h^*}^2+{\epsilon_3^*}^{-1}\left(1-2h^*\bar{\lambda}_0+{h^*}^2\overline{L}^2\right)}{8h^*\bar{\lambda}_0-4{h^*}^2(\epsilon_3^*+\overline{L}^2)}\right)^{\frac{1}{2}}+\frac{1+2h^*d^*}{2}<K+\frac{1}{2}.$$

This together with (3.15) implies

$$\lim_{\gamma\to 1} M_2(h^*,\gamma,\epsilon_3^*) = h^*\sqrt{N}\overline{L}^2\left(\frac{h^{*2}+\epsilon_3^{*-1}\left(1-2h^*\bar{\lambda}_0+h^{*2}\overline{L}^2\right)}{8h^*\bar{\lambda}_0-4h^{*2}(\epsilon_3^*+\overline{L}^2)}\right)^{\frac{1}{2}}+\frac{1+2h^*d^*}{2}.$$

Thus, there is $\gamma^*\in(\rho_{h^*,\epsilon_3^*}^{\frac{1}{2}},1)$ such that $M_2(h^*,\gamma^*,\epsilon_3^*)<K+\frac{1}{2}$. Therefore, by Theorem 3.12 we get the results of Theorem 3.13. $\square$

Remark 3.14. Theorem 3.13 implies that under the directed fixed communication topology, by properly choosing the quantizer and gain parameter we can always design a distributed protocol for each agent to realize an exponential average consensus with merely 1-bit data communication at each time step for each pair of adjacent agents.

4. Numerical example. In this section, we will give a numerical example to illustrate the results of section 3.

Example 4.1. Consider a network of three agents with the directed communication graph $\mathcal{G}=\{\mathcal{V}=\{1,2,3\},\mathcal{E}_{\mathcal{G}},\mathcal{A}_{\mathcal{G}}=[a_{ij}]_{3\times 3}\}$, where $\mathcal{E}_{\mathcal{G}}=\{(1,2),(2,1),(1,3),(3,1)\}$, $a_{12}=0.8$, $a_{21}=a_{31}=a_{23}=0.4$, and $a_{ij}=0$ if $(i,j)\notin\mathcal{E}_{\mathcal{G}}$. The initial states of the agents are given by $x_1(0)=2$, $x_2(0)=4$, and $x_3(0)=-3$. Set $K=1$. By Theorem 3.13 we can choose $h=0.0165$, $\gamma=0.994$, and $\epsilon_3=0.05$ such that $(h,\gamma,\epsilon_3)\in\Omega_2(K)$. g_0 is taken as 2.6667, which satisfies (3.16). The evolution of the states under the protocol (3.11)–(3.13) is shown in Figure 4.1. It can be seen that under the directed fixed topology $\mathcal{G}$ the average consensus is achieved asymptotically by merely 1-bit data exchange between each pair of adjacent agents at each time step.

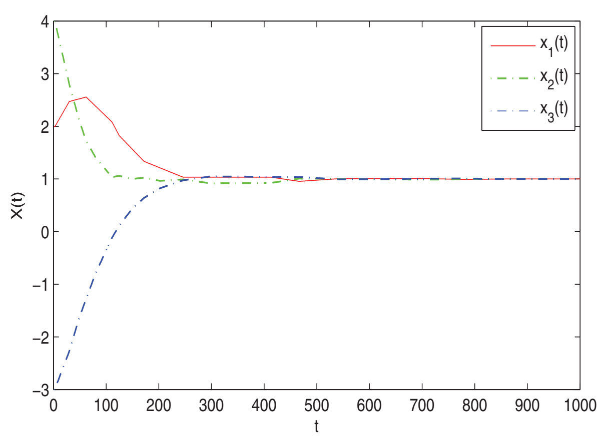

FIG. 4.1. *Curves of states under directed fixed topology when $K=1$.*

When the communication topologies of the agents are dynamically changing, we consider the time-varying communication graph $\mathcal{G}(t)=\{\mathcal{V},\mathcal{E}_{\mathcal{G}(t)},\mathcal{A}_{\mathcal{G}(t)}=[a_{ij}(t)]_{3\times 3}\}$, where $\mathcal{E}_{\mathcal{G}(t)}=\{(1,2),(2,1)\}$, $a_{12}(t)=a_{21}(t)=0.8$, $a_{ij}(t)=0$ if $(i,j)\notin\mathcal{E}_{\mathcal{G}(t)}$ when $t=2k$, $k=0,1,\ldots$, and where $\mathcal{E}_{\mathcal{G}(t)}=\{(1,3),(3,1)\}$, $a_{13}(t)=a_{31}(t)=0.8$, $a_{ij}(t)=0$ if $(i,j)\notin\mathcal{E}_{\mathcal{G}(t)}$ when $t=2k+1$, $k=0,1,\ldots$. It can be seen that $\mathcal{G}(t)$ is balanced and $\mathcal{G}_t^2=\sum_{i=t}^{t+1}\mathcal{G}(i)$ has a spanning tree. The initial states of agents are given by $x_1(0)=0.5$, $x_2(0)=0.8$, and $x_3(0)=-0.4$. By Theorem 3.10, for $K=3$, we can choose $h=0.0017$, $\gamma=0.9994$, $\epsilon_1=2.302\times 10^{-8}$, and $\epsilon_2=2$ such that

$(h, \gamma, \epsilon_1, \epsilon_2) \in \Omega_1(K)$. g_0 is taken as 0.4693, which satisfies (3.8). The evolution of the states under the protocol (3.1)–(3.3) is shown in Figure 4.2. It can be seen that under the time-varying topology sequence $\{\mathcal{G}(t), t = 0, 1, \ldots\}$ the average consensus is achieved asymptotically.

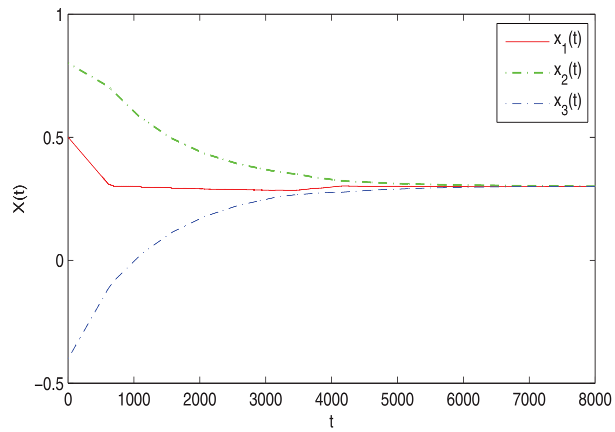

FIG. 4.2. *Curves of states under directed time-varying topologies when $K = 3$.*

5. Concluding remarks. This paper has considered the average consensus of multiagent systems with digital communication channels under directed time-varying communication topologies. Encoder-decoder schemes based on uniform quantizers with scaling are designed for the communications between each pair of agents. Distributed consensus protocols suitable for the time-varying topology are developed, and the convergence properties of the closed-loop systems are analyzed. The key idea of the designed communication scheme is to use a communication feedback to keep the internal state of the encoder at the sender agent consistent with the output of the decoder at the receiver agent, so that the quantization communication error of the state can be compensated. It is shown that for a periodically connected directed dynamic network and any uniform quantizers with quantization levels bigger than a certain finite constant, the MASs can achieve consensus exponentially provided the gain parameter and the scaling function are properly chosen.

In future work, it is worth considering how to weaken assumption (A2) to make the designed protocols suitable for a larger class of time-varying topology sequences and ensure simultaneously the use of as few communication data rates as possible. In addition, the quantized dynamic consensus problem and the cases for higher order MASs over a random switching topology and noisy digital communication channel, with additional considerations about asynchronous and time-delay communication, may also be considered.

Appendix A. Proof of Theorem 3.8. We prove the theorem by the following three steps.

Step 1: We will transform the coordinate to be the consensus error. Notice that $\mathbf{1}^T[(\mathcal{L}_{\mathcal{G}(t)} \odot \Lambda(t)) - (\mathcal{L}_{\mathcal{G}(t)} \odot \Lambda(t))^T]\mathbf{1} = 0$, and $\mathcal{G}(t)$ is balanced. Then, by (3.5) we have

$$\frac{1}{N}\sum_{i=1}^{N} x_i(t+1) = \frac{1}{N}\sum_{i=1}^{N} x_i(t) = \cdots = \frac{1}{N}\sum_{i=1}^{N} x_i(0). \tag{A.1}$$

This together with $\mathcal{L}_{\mathcal{G}(t)}\mathbf{1} = 0$ implies that the closed-loop system (3.5) can be transformed into the following form:

$$\delta(t+1) = \left(I - h\mathcal{L}_{\mathcal{G}(t)}\right)\delta(t) - h\left[\left(\mathcal{L}_{\mathcal{G}(t)} \odot \Lambda(t)\right) - \left(\mathcal{L}_{\mathcal{G}(t)} \odot \Lambda(t)\right)^T\right]\mathbf{1}, \tag{A.2}$$

where $\delta(t)$ and $\Lambda(t)$ are defined as in (3.4) and (3.6), respectively. Substituting (3.1) and (3.5) into (3.6) leads to the following recursive expression for $\lambda_{ij}(t)$, $i = 1, \dots, N$:

$$\lambda_{ij}(t+1) = \begin{cases} M_{ij}^{\lambda}(t) - g(t)q\left(g(t)^{-1}M_{ij}^{\lambda}(t)\right) & \text{if } j \in N_i^+(t+1), \\ M_{ij}^{\lambda}(t) & \text{if } j \in N_i^+ \setminus N_i^+(t+1), \\ 0 & \text{otherwise}, \quad t = 0, 1, \dots, \end{cases} \tag{A.3}$$

where $M_{ij}^{\lambda}(t) = \lambda_{ij}(t) + h\left[(\mathcal{L}_{\mathcal{G}(t)} \odot \Lambda(t)) - (\mathcal{L}_{\mathcal{G}(t)} \odot \Lambda(t))^T\right]\mathbf{1}\mid_j + h\mathcal{L}_{\mathcal{G}(t)}\delta(t)\mid_j$.

Let $w(t) = g^{-1}(t)\delta(t)$ and $z_{ij}(t) = g^{-1}(t)\lambda_{ij}(t)$. Then, from (A.2) and (A.3) we get

$$w(t+1) = \gamma^{-1}(I - h\mathcal{L}_{\mathcal{G}(t)})w(t) - \gamma^{-1}h\left[(\mathcal{L}_{\mathcal{G}(t)} \odot Z(t)) - (\mathcal{L}_{\mathcal{G}(t)} \odot Z(t))^T\right]\mathbf{1}, \tag{A.4}$$

$$z_{ij}(t+1) = \begin{cases} \gamma^{-1}\left(M_{ij}^{z}(t) - q(M_{ij}^{z}(t))\right) & \text{if } j \in N_i^+(t+1), \\ \gamma^{-1}M_{ij}^{z}(t) & \text{if } j \in N_i^+ \setminus N_i^+(t+1), \\ 0 & \text{otherwise}, \end{cases} \tag{A.5}$$

where $M_{ij}^{z}(t) = z_{ij}(t) + h\left[(\mathcal{L}_{\mathcal{G}(t)} \odot Z(t)) - (\mathcal{L}_{\mathcal{G}(t)} \odot Z(t))^T\right]\mathbf{1}\mid_j + h\mathcal{L}_{\mathcal{G}(t)}w(t)\mid_j$.

Step 2: We will prove that no quantizer is saturated. By (3.1) and (3.2) we have $\hat{x}_{ij}(0) = 0$, $j \in N_i^+$, $i = 1, \dots, N$. From (3.8) and assumption (A3) we know that

$$\begin{aligned} \left|M_{ij}^{z}(0)\right| &\le \frac{1}{g_0}\|X(0)\|_\infty + \frac{h}{g_0}\|\mathcal{L}_{\mathcal{G}(0)}\|_\infty \cdot \|\delta(0)\|_\infty + h\|(\mathcal{L}_{\mathcal{G}(0)} \odot Z(0))\mathbf{1}\|_\infty \\ &\quad + h\|(\mathcal{L}_{\mathcal{G}(0)} \odot Z(0))^T\mathbf{1}\|_\infty \le \frac{1}{g_0}\left(C_x + 2hd^*C_\delta + 2hd^*C_x\right) < K + \frac{1}{2}. \end{aligned}$$

Thus, no quantizer is saturated at the initial time. Suppose that at time $k = 0, 1, \dots, t$, no quantizer is saturated. Then, we can show that no quantizer is saturated at time $t+1$; i.e., for any $j \in N_i^+$, $i = 1, \dots, N$,

$$\begin{aligned} \left|M_{ij}^{z}(t+1)\right| &= \left|z_{ij}(t+1) + h\left[(\mathcal{L}_{\mathcal{G}(t+1)} \odot Z(t+1)) - (\mathcal{L}_{\mathcal{G}(t+1)} \odot Z(t+1))^T\right]\mathbf{1}\mid_j\right. \\ &\quad \left. + h\mathcal{L}_{\mathcal{G}(t+1)}w(t+1)\mid_j\right| < K + \frac{1}{2}. \end{aligned} \tag{A.6}$$

By direct computations we can get $\mathcal{L}_{\mathcal{G}(k)} \odot Z(k) = \bar{Z}(k) = [\bar{z}_{ij}(k)]$, where

$$\bar{z}_{ij}(k) = \begin{cases} a_{ij}(k)z_{ij}(k) & \text{if } j \in N_i^+(k), \\ 0 & \text{otherwise}, \quad k = 1, \dots, t+1. \end{cases}$$

Thus, by (A.5) we have $|\bar{z}_{ij}(k)| \le \frac{a_{ij}(k)}{2\gamma}$ and

$$\left\|(\mathcal{L}_{\mathcal{G}(k)} \odot Z(k))\mathbf{1}\right\|_\infty \le \frac{d^*}{2\gamma}, \quad \left\|(\mathcal{L}_{\mathcal{G}(k)} \odot Z(k))^T\mathbf{1}\right\|_\infty \le \frac{d^*}{2\gamma}, \quad k = 1, \dots, t+1. \tag{A.7}$$

For any positive integer m, by (A.4) we have

$$w((m+1)h_0) = \Phi((m+1)h_0, mh_0)w(mh_0) + \sum_{j=mh_0}^{(m+1)h_0-1} \gamma^{-1}h\Phi((m+1)h_0-1, j) \cdot \left[(\mathcal{L}_{\mathcal{G}(j)} \odot Z(j))^T - (\mathcal{L}_{\mathcal{G}(j)} \odot Z(j))\right]\mathbf{1},$$

where $\Phi(n+1,i) = \gamma^{-1}(I - h\mathcal{L}_{\mathcal{G}(n)})\Phi(n,i)$, $\Phi(i,i) = I$. Thus,

$$\begin{aligned}\|w((m+1)h_0)\|^2 =& w^T(mh_0)\Phi^T((m+1)h_0, mh_0)\Phi((m+1)h_0, mh_0)w(mh_0)\\ &+ 2w^T(mh_0)\Phi^T((m+1)h_0, mh_0)\sum_{j=mh_0}^{(m+1)h_0-1}\Phi((m+1)h_0-1, j)\\ &\cdot \gamma^{-1}h\left[(\mathcal{L}_{\mathcal{G}(j)} \odot Z(j))^T - (\mathcal{L}_{\mathcal{G}(j)} \odot Z(j))\right]\mathbf{1} + \gamma^{-2}h^2 I_{mh_0}\\ \triangleq& I_1 + I_2 + \gamma^{-2}h^2 I_{mh_0},\end{aligned} \tag{A.8}$$

where $I_{mh_0} = \sum_{j=mh_0}^{(m+1)h_0-1}\mathbf{1}^T\left[(\mathcal{L}_{\mathcal{G}(j)} \odot Z(j)) - (\mathcal{L}_{\mathcal{G}(j)} \odot Z(j))^T\right]\Phi^T((m+1)h_0-1, j)\cdot \sum_{k=mh_0}^{(m+1)h_0-1}\Phi((m+1)h_0-1, k)\left[(\mathcal{L}_{\mathcal{G}(k)} \odot Z(k))^T - (\mathcal{L}_{\mathcal{G}(k)} \odot Z(k))\right]\mathbf{1}$.

Since $\mathcal{G}(t)$, $t = 0, 1, \ldots,$ is balanced, by [22, Theorem 7], we have $\frac{\mathcal{L}_{\mathcal{G}(t)} + \mathcal{L}^T_{\mathcal{G}(t)}}{2} = \mathcal{L}_{\hat{\mathcal{G}}(t)}$, where $\hat{\mathcal{G}}(t)$ is the mirror graph of $\mathcal{G}(t)$. Thus, we have $\frac{\mathcal{L}_{\mathcal{G}(i)} + \mathcal{L}^T_{\mathcal{G}(i)}}{2} = \sum_{i=mh_0}^{(m+1)h_0-1}\mathcal{L}_{\hat{\mathcal{G}}(i)} = \mathcal{L}_{\sum_{i=mh_0}^{(m+1)h_0-1}\hat{\mathcal{G}}(i)} = \mathcal{L}_{\hat{\mathcal{G}}_{mh_0}^{(m+1)h_0-1}}$, which together with the condition $\inf_{m\geq 0}\lambda_{mh_0}^{h_0} \geq \lambda_0 > 0$ gives

$$\begin{aligned}&\left\|\Phi^T((m+1)h_0, mh_0)\Phi((m+1)h_0, mh_0)\right\|\\ &\leq \gamma^{-2h_0}\left\{\left\|I - 2h\sum_{i=mh_0}^{(m+1)h_0-1}\left(\frac{\mathcal{L}_{\mathcal{G}(i)} + \mathcal{L}^T_{\mathcal{G}(i)}}{2}\right)\right\| + \sum_{l=2}^{2h_0} h^l C^l_{2h_0}\left(\sup_{t\geq 0}\|\mathcal{L}_{\mathcal{G}(t)}\|\right)^l\right\}\\ &\leq \gamma^{-2h_0}\left\{1 - 2h\lambda_0 + \sum_{l=2}^{2h_0} h^l C^l_{2h_0} L^l\right\}.\end{aligned} \tag{A.9}$$

For any $\epsilon_1 > 0$, by $2x^Ty \leq \epsilon_1 x^Tx + \epsilon_1^{-1}y^Ty$ for all $x, y \in \mathbb{R}^N$, we have

$$\begin{aligned}I_2 &\leq \epsilon_1 h^2\gamma^{-2h_0}\|w(mh_0)\|^2 + \epsilon_1^{-1}\gamma^{-2}\left(1 - 2h\lambda_0 + \sum_{l=2}^{2h_0} h^l C^l_{2h_0}\left(\sup_{t\geq 0}\|\mathcal{L}_{\mathcal{G}(t)}\|\right)^l\right)I_{mh_0}\\ &\leq \epsilon_1 h^2\gamma^{-2h_0}\|w(mh_0)\|^2 + \epsilon_1^{-1}\gamma^{-2}\left(1 - 2h\lambda_0 + \sum_{l=2}^{2h_0} h^l C^l_{2h_0} L^l\right)I_{mh_0},\end{aligned} \tag{A.10}$$

where I_2 and I_{mh_0} are given as in (A.8). Noticing (A.7), by direct computations we can get $I_{mh_0} \leq \frac{Nd^{*2}}{\gamma^2}\sum_{j=0}^{2h_0-2}(h_0 - |j - (h_0-1)|)\gamma^{j-2}\sum_{l=0}^{j} h^l C^l_j L^l$, $m \geq 1$. This together with (A.8), (A.9), and (A.10) renders

(A.11)

$$\begin{aligned}&\|w((m+1)h_0)\|^2\\&\quad\le\left(\frac{\rho_{h,\epsilon_1}}{\gamma^{2h_0}}\right)\|w(mh_0)\|^2+\left[\gamma^{-2}h^2+\epsilon_1^{-1}\gamma^{-2}\left(1-2h\lambda_0+\sum_{l=2}^{2h_0}h^lC_{2h_0}^lL^l\right)\right]\cdot I_{mh_0}\\&\quad\le\left(\frac{\rho_{h,\epsilon_1}}{\gamma^{2h_0}}\right)^m\|w(h_0)\|^2+\frac{Nd^{*2}}{\gamma^2}\left[\gamma^{-2}h^2+\epsilon_1^{-1}\gamma^{-2}\left(1-2h\lambda_0+\sum_{l=2}^{2h_0}h^lC_{2h_0}^lL^l\right)\right]\\&\qquad\cdot\sum_{j=0}^{2h_0-2}(h_0-|j-(h_0-1)|)\gamma^{j-2}\sum_{l=0}^{j}C_j^lh^lL^l\cdot\frac{1-(\frac{\rho_{h,\epsilon_1}}{\gamma^{2h_0}})^m}{1-\frac{\rho_{h,\epsilon_1}}{\gamma^{2h_0}}},\end{aligned}$$

where ρ_{h,ϵ_1} is defined by (3.7). In addition, by (A.4) we have

$$\begin{aligned}\|w(t+1)\|^2&\le\rho_{h,\epsilon_2}\gamma^{-2}\|w(t)\|^2+\gamma^{-2}\left[\epsilon_2^{-1}(1+2hL+h^2L^2)+h^2\right]\\&\quad\cdot\left\|\left[(\mathcal{L}_{\mathcal{G}(t)}\odot Z(t))-(\mathcal{L}_{\mathcal{G}(t)}\odot Z(t))^T\right]\mathbf{1}\right\|^2,\end{aligned}\tag{A.12}$$

where ρ_{h,ϵ_2} is defined as in (3.7). Thus, from (A.7), (A.12), and $\|(\mathcal{L}_{\mathcal{G}(0)}\odot Z(0))\mathbf{1}\|_\infty\le\frac{d^*C_x}{g_0}$, we have

$$\begin{aligned}\|w(h_0)\|^2\le{}&\frac{NC_\delta^2\rho_{h,\epsilon_2}^{h_0}}{g_0^2\gamma^{2h_0}}+\frac{4Nd^{*2}C_x^2\rho_{h,\epsilon_2}^{h_0}}{g_0^2\gamma^{2h_0+2}}\left[\epsilon_2^{-1}(1+2hL+h^2L^2)+h^2\right]\\&+\frac{Nd^{*2}(1-\gamma^{-2h_0}\rho_{h,\epsilon_2}^{h_0})}{\gamma^4(1-\gamma^{-2}\rho_{h,\epsilon_2})}\cdot\left[\epsilon_2^{-1}(1+2hL+h^2L^2)+h^2\right].\end{aligned}\tag{A.13}$$

For any given $t\ge 0$, define $m_t=\lfloor\frac{t}{h_0}\rfloor$. Then $0\le t-m_th_0\le h_0$. Using (3.8), (A.11), (A.12), (A.13), and $\gamma\in(\rho_{h,\epsilon_1}^{\frac{1}{2h_0}},1)$, we have

(A.14)

$$\begin{aligned}&\|w(t+1)\|^2\\&\quad\le\rho_{h,\epsilon_2}^{t+1-m_th_0}\gamma^{-2(t+1-m_th_0)}\|w(m_th_0)\|^2\\&\qquad+\sum_{i=0}^{t-m_th_0}\epsilon_2^{-1}\rho_{h,\epsilon_2}^{i+1}\gamma^{-2(i+1)}\left\|\left[(\mathcal{L}_{\mathcal{G}(t-i)}\odot Z(t-i))-(\mathcal{L}_{\mathcal{G}(t-i)}\odot Z(t-i))^T\right]\mathbf{1}\right\|^2\\&\quad\le\left\{\frac{NC_\delta^2\rho_{h,\epsilon_2}^{2h_0}}{g_0^2\gamma^{4h_0}}+\frac{4Nd^{*2}C_x^2\rho_{h,\epsilon_2}^{2h_0+1}}{\epsilon_2g_0^2\gamma^{4h_0+2}}+\frac{Nd^{*2}\rho_{h,\epsilon_2}^{h_0+1}(1-\gamma^{-2h_0}\rho_{h,\epsilon_2}^{h_0})}{\epsilon_2\gamma^{2h_0+4}(1-\gamma^{-2}\rho_{h,\epsilon_2})}\right\}\left(\frac{\rho_{h,\epsilon_1}}{\gamma^{2h_0}}\right)^{m_t-1}\\&\qquad+\frac{N\rho_{h,\epsilon_1}d^{*2}\rho_{h,\epsilon_2}^{h_0}}{\epsilon_1\gamma^{2h_0+4}}\sum_{j=0}^{2h_0-2}(h_0-|j-(h_0-1)|)\gamma^{j-2}\sum_{l=0}^{j}C_j^lh^lL^l\frac{1-(\frac{\rho_{h,\epsilon_1}}{\gamma^{2h_0}})^{m_t-1}}{1-\frac{\rho_{h,\epsilon_1}}{\gamma^{2h_0}}}\\&\qquad+\frac{\rho_{h,\epsilon_2}Nd^{*2}(1-\gamma^{-2h_0}\rho_{h,\epsilon_2}^{h_0})}{\epsilon_2\gamma^4(1-\gamma^{-2}\rho_{h,\epsilon_2})}\le M,\end{aligned}$$

where M is given by (3.7).

By the definition of $z_{ij}(t+1)$ in (A.5) and assumption (A2), we have

$$|z_{ij}(t+1)| \le \max\left\{\frac{1}{2\gamma}, g^{-1}(t+1)\left|\hat{x}_{ji}(\tau_{ji}^t) - x_j(t+1)\right|\right\}, \tag{A.15}$$

where $\tau_{ji}^t = \max\{t_1 \le t : j \in N_i^+(t_1)\}$ and $t-\tau_{ji}^t \le T_0$. Furthermore, by the definition of the decoder Ψ_{ji}, we have

$$\left|g^{-1}(\tau_{ji}^t - 1)\hat{x}_{ji}(\tau_{ji}^t) - g^{-1}(\tau_{ji}^t - 1)x_j(\tau_{ji}^t)\right| < \frac{1}{2}. \tag{A.16}$$

Similarly to (A.14) for $w(\tau_{ji}^t)$, by (3.8) and (A.1) we have

$$\left|g^{-1}(t+1)(x_j(t+1) - J_N X(0))\right| \le M^{\frac{1}{2}}, \quad \left|g^{-1}(\tau_{ji}^t)(x_j(\tau_{ji}^t) - J_N X(0))\right| \le M^{\frac{1}{2}},$$

where M is given by (3.7). Therefore, $\left|x_j(t+1) - x_j(\tau_{ji}^t)\right| \le M^{\frac{1}{2}}\left[g(t+1) + g(\tau_{ji}^t)\right]$. This together with (A.16) implies

$$\begin{aligned}
&g^{-1}(t+1)\left|\hat{x}_{ji}(\tau_{ji}^t) - x_j(t+1)\right| \\
&\quad = g^{-1}(t+1)\left|(\hat{x}_{ji}(\tau_{ji}^t) - x_j(\tau_{ji}^t)) - (x_j(t+1) - x_j(\tau_{ji}^t))\right| \\
&\quad \le g^{-1}(t+1)\left[\frac{1}{2}g(\tau_{ji}^t - 1) + M^{\frac{1}{2}}(g(t+1) + g(\tau_{ji}^t))\right] \\
&\quad \le M^{\frac{1}{2}} + M^{\frac{1}{2}} \cdot \gamma^{-(T_0+1)} + \frac{1}{2}\gamma^{-(T_0+2)}.
\end{aligned}$$

Thus, by (3.8), (A.7), (A.14), and (A.15) we have

$$\begin{aligned}
\left|M_{ij}^z(t+1)\right| &\le |z_{ij}(t+1)| + h\left\|\mathcal{L}_{\mathcal{G}(t+1)}w(t+1)\right\|_\infty \\
&\qquad + h\left\|\left[(\mathcal{L}_{\mathcal{G}(t+1)} \odot Z(t+1)) - (\mathcal{L}_{\mathcal{G}(t+1)} \odot Z(t+1))^T\right]\mathbf{1}\right\|_\infty \\
&\le \max\left\{\frac{1}{2\gamma}, M^{\frac{1}{2}} + M^{\frac{1}{2}} \cdot \gamma^{-(T_0+1)} + \frac{1}{2}\gamma^{-(T_0+2)}\right\} + \frac{hd^*}{\gamma} + 2hd^*M^{\frac{1}{2}} \\
&< \left\lfloor M_1(h,\gamma,\epsilon_1,\epsilon_2) - \frac{1}{2}\right\rfloor + \frac{3}{2} = K_1(h,\gamma,\epsilon_1,\epsilon_2) + \frac{1}{2} \le K + \frac{1}{2},
\end{aligned}$$

where M is defined as in (3.7). Thus, (A.6) is correct.

Step 3: We will prove the exponential consensus of (3.5). Noticing that $\|w(0)\|_\infty \le \frac{C_\delta}{g_0}$, by (A.14) we have $\sup_{t\ge 0}\|w(t)\|_\infty \le \max\{C_\delta/g_0, M^{\frac{1}{2}}\} < \infty$. By the definition of $w(t)$ and $\gamma \in (0,1)$ we get $\lim_{t\to\infty}\|\delta(t)\|_\infty = 0$. This together with (A.1) leads to the consensus of (3.5).

From (A.14), $\delta(t) = g_0\gamma^t w(t)$, and $\gamma \in (\rho_{h,\epsilon_1}^{\frac{1}{2h_0}}, 1)$, we have

$$\begin{aligned}
\limsup_{t\to\infty}\frac{\|\delta(t+1)\|^2}{\gamma^{2(t+1)}} \le g_0^2\Bigg\{&\frac{N\rho_{h,\epsilon_2}{d^*}^2(1-\gamma^{-2h_0}\rho_{h,\epsilon_2}^{h_0})}{\epsilon_2\gamma^4(1-\gamma^{-2}\rho_{h,\epsilon_2})} + \frac{N\rho_{h,\epsilon_1}{d^*}^2\rho_{h,\epsilon_2}^{h_0}}{\epsilon_1\gamma^4(\gamma^{2h_0} - \rho_{h,\epsilon_1})} \\
&\cdot\sum_{j=0}^{2h_0-2}(h_0 - |j-(h_0-1)|)\gamma^{j-2}\sum_{l=0}^{j}C_j^l h^l L^l\Bigg\}.
\end{aligned}$$

Thus, (3.9) is true. $\square$

Appendix B. Proof of Theorem 3.12. Let $w(t) = g^{-1}(t)\delta(t)$, $z(t) = g^{-1}(t)e(t)$. By $\mathcal{L}_{\mathcal{G}}\mathbf{1} = \mathbf{1}^T\mathcal{L}_{\mathcal{G}} = 0$ and $g(t) = g_0\gamma^t$, we can transform the closed-loop system (3.14) into the following form:

$$\begin{aligned} w(t+1) &= \gamma^{-1}(I - h\mathcal{L}_{\mathcal{G}})w(t) + \gamma^{-1}h\mathcal{L}_{\mathcal{G}}z(t), \\ z(t+1) &= \gamma^{-1}\Delta(t), \end{aligned} \tag{B.1}$$

where $\Delta(t) = (I + h\mathcal{L}_{\mathcal{G}})z(t) - h\mathcal{L}_{\mathcal{G}}w(t) - Q\left[(I + h\mathcal{L}_{\mathcal{G}})z(t) - h\mathcal{L}_{\mathcal{G}}w(t)\right]$.

We now prove that no quantizer is saturated. From (3.11) and (3.12) we know that $\hat{X}(0) = 0$. Noticing that $\mathcal{L}_{\mathcal{G}}\mathbf{1} = 0$, by (3.16) and assumption (A3) we have

$$\|(I + h\mathcal{L}_{\mathcal{G}})z(0) - h\mathcal{L}_{\mathcal{G}}w(0)\|_\infty = \left\|\frac{X(0)}{g_0}\right\|_\infty \le \frac{C_x}{g_0} < K + \frac{1}{2}.$$

Suppose that when $k = 0, 1, \ldots, t$, no quantizer is saturated, i.e., $\sup_{0\le k\le t}\|\Delta(k)\|_\infty \le \frac{1}{2}$. Then, by (B.1) we have

$$\sup_{1\le k\le t+1}\|z(k)\|_\infty \le \frac{1}{2\gamma}. \tag{B.2}$$

As $\mathcal{G}$ is balanced, by [22, Theorem 7] we have $\frac{\mathcal{L}_{\mathcal{G}}+\mathcal{L}_{\mathcal{G}}^T}{2} = \mathcal{L}_{\hat{\mathcal{G}}}$. From assumption (A4) it follows that $\hat{\mathcal{G}}$ is a strongly connected undirected graph, and hence $\lambda_2(\mathcal{L}_{\hat{\mathcal{G}}}) > 0$ [11]. Noticing that $2x^Ty \le \epsilon_3 x^Tx + \epsilon_3^{-1}y^Ty$ for all $x, y \in \mathbb{R}^N$, $\epsilon_3 > 0$, we have

$$\begin{aligned} \|w(t+1)\|^2 &= \gamma^{-2}\left[w^T(t)(I - 2h\mathcal{L}_{\hat{\mathcal{G}}} + h^2\mathcal{L}_{\mathcal{G}}^T\mathcal{L}_{\mathcal{G}})w(t)\right. \\ &\quad \left. + 2hw^T(t)(I - h\mathcal{L}_{\mathcal{G}}^T)\mathcal{L}_{\mathcal{G}}z(t) + h^2z^T(t)\mathcal{L}_{\mathcal{G}}^T\mathcal{L}_{\mathcal{G}}z(t)\right] \\ &\le \gamma^{-2}\Big[(1 - 2h\bar{\lambda}_0 + h^2\overline{L}^2 + \epsilon_3 h^2)\|w(t)\|^2 \\ &\quad + \epsilon_3^{-1}z^T(t)\mathcal{L}_{\mathcal{G}}^T(I - h\mathcal{L}_{\mathcal{G}})(I - h\mathcal{L}_{\mathcal{G}}^T)\mathcal{L}_{\mathcal{G}}z(t) + h^2z^T(t)\mathcal{L}_{\mathcal{G}}^T\mathcal{L}_{\mathcal{G}}z(t)\Big] \\ &\le \left(\frac{\rho_{h,\epsilon_3}}{\gamma^2}\right)\|w(t)\|^2 + \epsilon_3^{-1}\gamma^{-2}\rho_{h,\epsilon_3}\overline{L}^2\|z(t)\|^2. \end{aligned}$$

By $\rho_{h,\epsilon_3} \in (0,1)$, $\gamma \in (\rho_{h,\epsilon_3}^{\frac{1}{2}}, 1)$, (B.2), and assumption (A3), we have

(B.3)

$$\begin{aligned} \|w(t+1)\|^2 &\le \left(\frac{\rho_{h,\epsilon_3}}{\gamma^2}\right)^{t+1}\|w(0)\|^2 \\ &\quad + \left(\frac{\rho_{h,\epsilon_3}}{\gamma^2}\right)^{t+1}\epsilon_3^{-1}\overline{L}^2\|z(0)\|^2 + \sum_{i=0}^{t-1}\left(\frac{\rho_{h,\epsilon_3}}{\gamma^2}\right)^{i+1}\epsilon_3^{-1}\overline{L}^2\|z(t-i)\|^2 \\ &\le \left(\frac{\rho_{h,\epsilon_3}}{\gamma^2}\right)^t\left\{\frac{N\rho_{h,\epsilon_3}C_\delta^2}{g_0^2\gamma^2} + \frac{NC_x^2\overline{L}^2\rho_{h,\epsilon_3}}{\epsilon_3 g_0^2\gamma^2}\right\} + \frac{N\overline{L}^2\rho_{h,\epsilon_3}}{4\epsilon_3\gamma^2(\gamma^2 - \rho_{h,\epsilon_3})}\left(1 - \left(\frac{\rho_{h,\epsilon_3}}{\gamma^2}\right)^t\right) \\ &\le \max\left\{\frac{N\rho_{h,\epsilon_3}C_\delta^2}{g_0^2\gamma^2} + \frac{NC_x^2\overline{L}^2\rho_{h,\epsilon_3}}{\epsilon_3 g_0^2\gamma^2}, \frac{N\overline{L}^2\rho_{h,\epsilon_3}}{4\epsilon_3\gamma^2(\gamma^2 - \rho_{h,\epsilon_3})}\right\}. \end{aligned}$$

This together with (3.16) and (B.2) gives

$$\begin{aligned}
&\|(I + h\mathcal{L}_{\mathcal{G}})z(t+1) - h\mathcal{L}_{\mathcal{G}}w(t+1)\|_\infty \\
&\quad \le \|I + h\mathcal{L}_{\mathcal{G}}\|_\infty \|z(t+1)\|_\infty + h\|\mathcal{L}_{\mathcal{G}}\| \cdot \|w(t+1)\| \\
&\quad \le \frac{1+2hd^*}{2\gamma} + h\overline{L}\max\left\{ \left(\frac{N\rho_{h,\epsilon_3}C_\delta^2}{g_0^2\gamma^2} + \frac{NC_x^2\overline{L}^2\rho_{h,\epsilon_3}}{\epsilon_3 g_0^2\gamma^2} \right)^{\frac{1}{2}}, \frac{\sqrt{N}\overline{L}\rho_{h,\epsilon_3}^{\frac{1}{2}}}{2\gamma\epsilon_3^{\frac{1}{2}}(\gamma^2-\rho_{h,\epsilon_3})^{\frac{1}{2}}} \right\} \\
&\quad = M_2(h,\gamma,\epsilon_3) < \left\lfloor M_2(h,\gamma,\epsilon_3) - \frac{1}{2} \right\rfloor + \frac{3}{2} = K_2(h,\gamma,\epsilon_3) + \frac{1}{2} \le K + \frac{1}{2}.
\end{aligned}$$

Thus, no quantizer is saturated. Noticing that $\|w(0)\|_\infty \le \frac{C_\delta}{g_0}$, by (3.16) and (B.3) we have

$$\sup_{t\ge 0} \|w(t)\|_\infty \le \max\left\{ \frac{C_\delta}{g_0}, \frac{\sqrt{N}\overline{L}\rho_{h,\epsilon_3}^{\frac{1}{2}}}{2\gamma\epsilon_3^{\frac{1}{2}}(\gamma^2-\rho_{h,\epsilon_3})^{\frac{1}{2}}} \right\} < \infty.$$

Hence, by the definition of $w(t)$ and $\gamma \in (0,1)$ we get $\lim_{t\to\infty}\|\delta(t)\|_\infty = 0$. This together with $\mathbf{1}^T\mathcal{L}_{\mathcal{G}} = 0$ and $\frac{1}{N}\sum_{j=1}^N x_j(t+1) = \frac{1}{N}\sum_{j=1}^N x_j(t) = \cdots = \frac{1}{N}\sum_{j=1}^N x_j(0)$ leads to (3.17).

From (B.3), $\delta(t) = g_0\gamma^t w(t)$, and $\gamma \in (\rho_{h,\epsilon_3}^{\frac{1}{2}}, 1)$, we have

$$\begin{aligned}
\|\delta(t+1)\|^2 \le g_0^2\gamma^{2(t+1)} \left(\frac{\rho_{h,\epsilon_3}}{\gamma^2}\right)^t \left\{ \frac{N\rho_{h,\epsilon_3}C_\delta^2}{g_0^2\gamma^2} + \frac{NC_x^2\overline{L}^2\rho_{h,\epsilon_3}}{\epsilon_3 g_0^2\gamma^2} \right\} \\
+ \frac{Ng_0^2\gamma^{2(t+1)}\overline{L}^2\rho_{h,\epsilon_3}}{4\epsilon_3\gamma^2(\gamma^2-\rho_{h,\epsilon_3})} \left(1 - \left(\frac{\rho_{h,\epsilon_3}}{\gamma^2}\right)^t\right).
\end{aligned}$$

Thus, (3.17) is true. $\square$

REFERENCES

[1] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, *Randomized gossip algorithms*, IEEE Trans. Inform. Theory, 52 (2006), pp. 2508–2530.

[2] T. C. Aysal, M. J. Coates, and M. G. Rabbat, *Distributed average consensus with dithered quantization*, IEEE Trans. Signal Process., 56 (2008), pp. 4905–4918.

[3] R. Carli, F. Bullo, and S. Zampieri, *Quantized average consensus via dynamic coding/decoding schemes*, Internat. J. Robust Nonlinear Control, 20 (2010), pp. 156–175.

[4] R. Carli, G. Como, P. Frasca, and F. Garin, *Distributed averaging on digital erasure networks*, Automatica J. IFAC, 47 (2011), pp. 115–121.

[5] R. Carli, F. Fagnani, P. Frasca, and S. Zampieri, *A probabilistic analysis of the average consensus algorithm with quantized communication*, in Proceedings of the 17th IFAC World Congress, Seoul, Korea, 2008, pp. 8062–8067.

[6] R. Carli, F. Fagnani, A. Speranzon, and S. Zampieri, *Communication constraints in the average consensus problem*, Automatica J. IFAC, 44 (2008), pp. 671–684.

[7] F. Fagnani and S. Zampieri, *Average consensus with packet drop communication*, SIAM J. Control Optim., 48 (2009), pp. 102–133.

[8] J. A. Fax and R. M. Murray, *Information flow and cooperative control of vehicle formations*, IEEE Trans. Automat. Control, 49 (2004), pp. 1465–1476.

[9] P. Frasca, R. Carli, F. Fagnani, and S. Zampieri, *Average consensus on networks with quantized communication*, Internat. J. Robust Nonlinear Control, 19 (2009), pp. 1787–1816.

[10] M. Fu and L. Xie, *The sector bound approach to quantized feedback control*, IEEE Trans. Automat. Control, 50 (2005), pp. 1698–1711.

[11] C. D. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.

[12] L. Guo, *Time-Varying Stochastic Systems*, Jilin Science and Technology Press, Changchun, 1993 (in Chinese).

[13] M. Huang and J. H. Manton, *Stochastic consensus seeking with noisy and directed inter-agent communication: Fixed and randomly varying topologies*, IEEE Trans. Automat. Control, 55 (2010), pp. 235–241.

[14] A. Jadbabaie, J. Lin, and A. S. Morse, *Coordination of groups of mobile autonomous agents using nearest neighbor rules*, IEEE Trans. Automat. Control, 48 (2003), pp. 988–1001.

[15] S. Kar and J. M. F. Moura, *Distributed average consensus in sensor networks with quantized inter-sensor communication*, in Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing, Las Vegas, NV, 2008, pp. 2281–2284.

[16] A. Kashyap, T. Basar, and R. Srikant, *Quantized consensus*, Automatica J. IFAC, 43 (2007), pp. 1192–1203.

[17] T. Li, M. Fu, L. Xie, and J. F. Zhang, *Distributed consensus with limited communication data rate*, IEEE Trans. Automat. Control, 56 (2011), pp. 279–292.

[18] T. Li and J. F. Zhang, *Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises*, IEEE Trans. Automat. Control, 55 (2010), pp. 2043–2057.

[19] L. Moreau, *Stability of multiagent systems with time-dependent communication links*, IEEE Trans. Automat. Control, 50 (2005), pp. 169–182.

[20] A. Nedic, A. Olshevsky, A. Ozdaglar, and J. N. Tsitsiklis, *On distributed averaging algorithms and quantization effects*, IEEE Trans. Automat. Control, 54 (2009), pp. 2506–2517.

[21] R. Olfati-Saber, *Flocking for multi-agent dynamic systems: Algorithms and theory*, IEEE Trans. Automat. Control, 51 (2006), pp. 401–420.

[22] R. Olfati-Saber and R. M. Murray, *Consensus problems in networks of agents with switching topology and time-delays*, IEEE Trans. Automat. Control, 49 (2004), pp. 1520–1533.

[23] W. Ren and R. W. Beard, *Consensus seeking in multiagent systems under dynamically changing interaction topologies*, IEEE Trans. Automat. Control, 50 (2005), pp. 655–661.

[24] W. Ren, R. W. Beard, and D. B. Kingston, *Multi-agent Kalman consensus with relative uncertainty*, in Proceedings of the 2005 American Control Conference, Portland, OR, 2005, pp. 1865–1870.

[25] S. Tatikonda and S. Mitter, *Control under communication constraints*, IEEE Trans. Automat. Control, 49 (2004), pp. 1056–1068.

[26] J. N. Tsitsiklis, *Problems in Decentralized Decision Making and Computation*, Ph.D. thesis, Department of Electrical Engineering and Computer Science, MIT, Cambridge, MA, 1984.

[27] L. Xiao, S. Boyd, and S. Lall, *A scheme for robust distributed sensor fusion based on average consensus*, in Proceedings of the 4th International Symposium on Information Processing in Sensor Networks, Los Angeles, CA, 2005, pp. 63–70.